Published by Institute of Physics Publishing for SISSA/ISAS
Received: *March 12, 2002*
Accepted: *May 31, 2002*

Seiberg-Witten curve for the E-string theory

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Abstract: We construct the Seiberg-Witten curve for the E-string theory in six-dimensions. The curve is expressed in terms of affine E_8 characters up to level 6 and is determined by using the mirror-type transformation so that it reproduces the number of holomorphic curves in the Calabi-Yau manifold and the amplitudes of $\mathcal{N} = 4$ U(n) Yang-Mills theory on $\frac{1}{2}K_3$. We also show that our curve flows to known five- and four-dimensional Seiberg-Witten curves in suitable limits.

Keywords: Topological Field Theories, Global Symmetries, Supersymmetric Effective Theories, Differential and Algebraic Geometry.

JHEP05(2002)058

Contents

1. Introduction

E-string was first discovered in the study of zero-size instantons in heterotic string theory [1]. In the M-theory description [2] E-string appears at the intersection of the (end-of-world) 9-brane and a membrane which connects the 9-brane to a 5-brane representing the small instanton. An E-string carries a level one E_8 current algebra and possesses one half of the degrees of freedom of the heterotic string. In fact an E-string may be obtained by wrapping a M5-brane around $\frac{1}{2}K_3$ surface [3] while a heterotic string is obtained by wrapping M5-brane around K_3. The second homology $b_2^+ = 1$, $b_2^- = 9$ of $\frac{1}{2}K_3$ generates 1 (9) right-(left-)moving bosonic degrees of freedom in 2-dimensions. Together with the freedom representing the position of the M5-brane one obtains 4 (12) right (left) bosonic degrees of freedom of E-string which is one half of the heterotic string. E-string possesses one half of supersymmetry of the heterotic theory and the reduction of supersymmetry leads to threshold BPS bound states and non-trivial dynamics in low energy E-string theory.

It is known that the $\frac{1}{2}K_3$ can be obtained by blowing up 9 points of $\mathbf{CP}^2$ and is also called as an almost del Pezzo surface $\mathcal{B}_9$. Elements in the second homology group of $\frac{1}{2}K_3$ span a unimodular lattice $\Gamma^{9,1}$ with signature $(1,9)$ which contains the lattice Γ^8 of E_8, $\Gamma^{9,1} = \Gamma^8 \oplus \Gamma^{1,1}$. The existence of the lattice Γ^8 is the origin of the E_8 symmetry in E-string theory.

It is known that the del Pezzo surface $\mathcal{B}_9$ possesses an elliptic fibration over a $\mathbf{P}^1$ with 12 singular fibers. The fibration is described by a family of elliptic curves written in a Weierstrass form

$$\mathcal{C}: \ y^2 = 4x^3 - f(u,\tau;m_i)x - g(u,\tau;m_i)\,, \qquad i = 1,2,\ldots,8 \tag{1.1}$$

where the coefficient functions f, g are degree 4 and 6 polynomials in u, respectively

$$f(u,\tau;m_i) = a_0(\tau)u^4 + a_1(\tau;m_i)u^3 + a_2(\tau;m_i)u^2 + a_3(\tau;m_i)u + a_4(\tau;m_i)\,, \tag{1.2}$$

$$\begin{aligned} g(u,\tau;m_i) = {} & b_0(\tau)u^6 + b_1(\tau;m_i)u^5 + b_2(\tau;m_i)u^4 + b_3(\tau;m_i)u^3 + b_4(\tau;m_i)u^2 + \\ & + b_5(\tau;m_i)u + b_6(\tau;m_i)\,. \end{aligned} \tag{1.3}$$

u parametrizes the base $\mathbf{P}^1$ and m_i $i = 1,2,\ldots,8$ represent the position of the blow-up points (position of the 9th blow-up point is fixed by the other 8 points). τ gives the modulus of the elliptic fiber at $u = \infty$ and is also the modulus of the torus T^2 upon which E-string theory is compactified down to 4-dimensions. $\{m_i\}$ correspond physically to the masses of 8 hypermultiplets in $\mathcal{N} = 2$ SU(2) gauge theory on $\mathbf{R}^4 \times T^2$.

With τ being the modulus at $u = \infty$, $a_0(\tau), b_0(\tau)$ are fixed to

$$a_0(\tau) = \frac{1}{12}E_4(\tau)\,, \qquad b_0(\tau) = \frac{1}{216}E_6(\tau)\,, \tag{1.4}$$

where E_n is the Eisenstein series with weight n,

$$E_{2n}(\tau) = 1 + \frac{(2\pi i)^{2n}}{(2n-1)!\,\zeta(2n)} \sum_{m=1} \sigma_{2n-1}(m)q^m\,, \qquad \sigma_k(m) = \sum_{d|m} d^k\,, \qquad q = e^{2\pi i\tau}\,. \tag{1.5}$$

By shifting the variable u we can eliminate either one of the functions a_1 or b_1. In the following we choose a "gauge" where we put

$$a_1(\tau;m_i) = 0\,. \tag{1.6}$$

We call eq. (1.1) as the six-dimensional Seiberg-Witten (SW) curve: the six-dimensional curve determines the prepotential of E-string theory and gives a complete description of its low-energy dynamics. Previously there were attempts at deriving the six-dimensional curve [4]–[7], however, only partial results with a few non-vanishing mass parameters $\{m_i\}$ were obtained. In this paper we will determine all the coefficient functions a_j, b_j and the curve (1.1) for arbitrary values of $\{m_i\}$. It turns out that the functions a_j, b_j are expressed in terms of characters of affine E_8 at level j.

Six-dimensional curves with lower-rank symmetry groups can be easily obtained from (1.1) by taking some of the mass parameters $\{m_i\}$ to ∞ or adjusting them to special values. We shall show that the known five-dimensional SW curves [6] are reproduced from the six-dimensional curve by taking the limit $\mathrm{Im}\tau \to +\infty$. Four-dimensional curves [8, 9, 10] are also obtained by further taking the remaining period of T^2 to 0. Thus our six-dimensional curve serves as some kind of master theory encompassing all possible SW theories.

Let us now recall the relation between BPS states of the E-string and holomorphic curves in the Calabi-Yau 3-fold [11] and also the partition function of the $\mathcal{N}=4$ U(n) Yang-Mills theory on $\frac{1}{2}K_3$ [3]. Consider an F-theory compactification down to 6-dimensions on a Calabi-Yau 3-fold K which is elliptically fibered over a base B. We choose a curve C inside the base B so that the elliptic fibration restricted to C gives the $\frac{1}{2}K_3$. Consider a D3-brane wrapped around C, which gives rise to a string in 6-dimensions. Low energy dynamics of such a string can be studied by looking at its BPS spectrum. We compactify the 5th dimension on a circle R and count the BPS states of the string with winding number n and momentum k. We denote the number of these states as $Z_{n,k}$.

Next by using a duality between F- and M-theory we go to a M-theory description: a compactification of F-theory on $K \times S^1$, where the S^1 has radius R, is equivalent to a compactification of M-theory on K with the Kähler parameter of the elliptic fiber being equal to $1/R$. In M-theory 5-dimensional BPS states are obtained by wrapping a membrane around holomorphic curves. Then the number of BPS states $Z_{n,k}$ is given by the number of curves in the class $n[C]+k[E]$ where $[C]$ ($[E]$) denotes the class of the base (elliptic fiber) of $\frac{1}{2}K_3$. Thus the counting of BPS states of 6-dimensional E-string wrapped around a circle is related to the counting of holomorphic curves in K which can be analyzed by using the technique of mirror symmetry.

If we introduce M5-branes, the number of BPS states becomes further related to the partition function of U(n) Yang-Mills theory on $\frac{1}{2}K_3$ [3]. Consider a M5-brane wrapped around $\frac{1}{2}K_3$. One then obtains a string in 5-dimensions. In order to study its spectrum we may wrap the string around a circle and compute its toroidal partition function. Then the M5-brane becomes effectively wrapped around $\frac{1}{2}K_3 \times T^2$. If we consider the string n-times wound around a circle, M5-brane wraps around T^2 n-times and one obtains $\mathcal{N}=4$ U(n) Yang-Mills theory on $\frac{1}{2}K_3$. The gauge coupling constant is given by the modulus τ of the torus and the momentum k is mapped to the instanton number. Thus $Z_{n,k}$ is also interpreted as the k-instanton contribution to the partition function of U(n) gauge theory on $\frac{1}{2}K_3$.

We want to determine the prepotential of the theory

$$
\begin{aligned}
\mathcal{F}(\phi,\tau;m_i) &= \mathcal{F}_{\text{classical}} - \frac{i}{8\pi^3}\sum_{n=1} q^{n/2} Z_n(\tau;m_i) e^{2\pi i n\phi}\,, \\
q^{n/2} Z_n(\tau;m_i) &= \sum_{k=0} Z_{n,k}(m_i) q^k\,, \\
q &= e^{2\pi i\tau}\,,
\end{aligned}
\tag{1.7}
$$

where ϕ is the Kähler parameter which is dual (mirror) to the variable u. We have inserted a factor $q^{n/2}$ for the sake of a good modular property of Z_n: Z_n becomes a (quasi-)Jacobi form of weight -2.

Actually the functions Z_n have already been computed recursively up to Z_4 by making use of the holomorphic anomaly and the gap condition ($Z_{n,k} = 0$ for $0 < k < n$) in [3]. One may continue this computation and obtain more data on Z_n. On the other hand, from the SW curve (1.1) one can compute the prepotential by making use of the mirror-type transformation and express Z_n in terms of the coefficient functions a_j, b_j. It turns out that Z_n is written as a polynomial in a_j, b_j, $0 \leq j \leq n$ and in particular linear in a_n, b_n. Hence given the data $\{Z_n\}$ it is easy to determine $\{a_j, b_j\}$. In our actual computation we have used the data up to Z_8 to determine the Seiberg-Witten curve.

2. Instanton expansion

Let us next describe the standard mirror-type transformation in Seiberg-Witten theory adopted to the present situation [12]. The coupling constant $\tilde{\tau}$ of $\mathcal{N} = 2$ gauge theory is given by the modulus of the elliptic curve

$$j(\tilde{\tau}) = \frac{1728 f(u,\tau;m_i)^3}{f(u,\tau;m_i)^3 - 27 g(u,\tau;m_i)^2}\,. \tag{2.1}$$

Here j is the elliptic j-function. We may expand the right-hand side of the above equation in $1/u$ and obtain

$$\begin{aligned} j(\tilde{\tau}) &= \frac{1728 E_4(\tau)^3}{E_4(\tau)^3 - E_6(\tau)^2} + \frac{1}{4}\frac{E_4(\tau)^3 E_6(\tau) b_1(\tau;m_i)}{\Delta(\tau)^2}\frac{1}{u} + \cdots \\ &= j(\tau) + \frac{1}{4}\frac{E_4(\tau)^3 E_6(\tau) b_1(\tau;m_i)}{\Delta(\tau)^2}\frac{1}{u} + \cdots\,, \end{aligned} \tag{2.2}$$

where $\Delta = ({E_4}^3 - {E_6}^2)/1728 = \eta^{24}$. Thus $\tilde{\tau} = \tau$ at $u = \infty$. By inverting the j-function in (2.2) we can generate a Taylor series expansion of $\tilde{\tau}$ around $u = \infty$,

$$\tilde{\tau} = \tau + \frac{i}{8\pi}\frac{E_4 b_1}{\Delta}\frac{1}{u} + \frac{i}{384\pi}\frac{-4E_6\Delta a_2 + 48 E_4 \Delta b_2 + 5 E_4 E_6 {b_1}^2 + E_2 {E_4}^2 {b_1}^2}{\Delta^2}\frac{1}{u^2} + \cdots \tag{2.3}$$

Periods of the torus ω_1, ω_2 are given by

$$\omega_1 = \frac{i}{2\pi}\left(\frac{E_4(\tilde{\tau})}{12 f(u,\tau;m_i)}\right)^{\frac{1}{4}}, \qquad \omega_2 = \tilde{\tau}\omega_1\,. \tag{2.4}$$

Both of these ω_i are also expanded in Taylor series in $1/u$. We can then easily integrate them and obtain functions ϕ, ϕ_D

$$\phi = \int du\, \omega_1(u,\tau;m_i)\,, \qquad \phi_D = \int du\, \omega_2(u,\tau;m_i) = \int du\, \tilde{\tau}(u,\tau;m_i)\omega_1(u,\tau;m_i)\,. \tag{2.5}$$

Since $\partial\phi/\partial u = \omega_1$, $\partial\phi_D/\partial u = \omega_2$, ϕ, ϕ_D correspond to the variables of the Coulomb branch a, a_D in the 4-dimensional Seiberg-Witten theory. Prepotential is defined by

$$\frac{\partial\mathcal{F}}{\partial\phi} = \phi_D \tag{2.6}$$

as usual.

Lower order computation goes as follows: ω_1 has an expansion

$$\omega_1 = \frac{i}{2\pi}\frac{1}{u} - \frac{i}{96\pi}\frac{(E_2E_4 - E_6)b_1}{\Delta}\frac{1}{u^2} + \cdots \tag{2.7}$$

ϕ is then given by

$$\phi = \phi_0 + \frac{i}{2\pi}\ln u + \frac{i}{96\pi}\frac{(E_2E_4 - E_6)b_1}{\Delta}\frac{1}{u} + \cdots \tag{2.8}$$

where ϕ_0 is an integration constant. Thus

$$\frac{1}{u} = e^{2\pi i(\phi-\phi_0)} + \frac{1}{48}\frac{(E_2E_4 - E_6)b_1}{\Delta}e^{4\pi i(\phi-\phi_0)} + \cdots \tag{2.9}$$

and therefore

$$\begin{aligned}\frac{\partial^2\mathcal{F}}{\partial\phi^2} = \tilde{\tau} = \tau + \frac{i}{8\pi}\frac{E_4b_1}{\Delta}e^{2\pi i(\phi-\phi_0)} + \\ + \frac{i}{192\pi}\frac{-2E_6\Delta a_2 + 24E_4\Delta b_2 + 2E_4E_6{b_1}^2 + E_2{E_4}^2{b_1}^2}{\Delta^2}e^{4\pi i(\phi-\phi_0)} + \cdots\end{aligned} \tag{2.10}$$

By integrating twice in ϕ we obtain the prepotential. In the following we choose $\phi_0 = -(\ln\eta^{12})/2\pi i - 1/2$.

We have performed the above transformation up to higher orders and obtained the partition functions expressed in terms of the coefficient functions of the SW curve. We present lower order terms

$$Z_1 = -\frac{1}{4}\frac{E_4}{\eta^{12}}b_1\,, \tag{2.11}$$

$$Z_2 = \frac{1}{8}\left(-\frac{1}{24}E_6\,a_2 + \frac{1}{2}E_4\,b_2 + \frac{1}{24}\frac{E_4\,E_6}{\eta^{24}}{b_1}^2 + \frac{1}{48}\frac{E_2\,{E_4}^2}{\eta^{24}}{b_1}^2\right), \tag{2.12}$$

$$\begin{aligned}Z_3 = \frac{1}{27}\Bigg(&\frac{1}{16}\eta^{12}E_6\,a_3 - \frac{3}{4}\eta^{12}E_4\,b_3 - \frac{9}{2}\eta^{12}a_2\,b_1 + \frac{5}{512}\frac{{E_4}^3}{\eta^{12}}a_2\,b_1 + \\ &+ \frac{3}{512}\frac{E_2\,E_4\,E_6}{\eta^{12}}a_2\,b_1 - \frac{15}{128}\frac{E_4\,E_6}{\eta^{12}}b_2\,b_1 - \frac{9}{128}\frac{E_2\,{E_4}^2}{\eta^{12}}b_2\,b_1 + \\ &+ \frac{591}{64}\frac{E_4}{\eta^{12}}{b_1}^3 - \frac{17}{2048}\frac{{E_4}^4}{\eta^{36}}{b_1}^3 - \frac{3}{512}\frac{E_2\,{E_4}^2\,E_6}{\eta^{36}}{b_1}^3 - \frac{3}{2048}\frac{{E_2}^2\,{E_4}^3}{\eta^{36}}{b_1}^3\Bigg).\end{aligned} \tag{2.13}$$

We have obtained similar formulas up to Z_8.

3. Holomorphic anomaly and gap condition

Partition functions Z_n of U(n) gauge theory on $\frac{1}{2}K_3$ have been computed in [3] by imposing the relation of holomorphic anomaly and gap condition (see [13] for a different analysis). Holomorphic anomaly occurs in the gauge theory due to the appearance of reducible connections and corresponds to the existence of threshold bound states in the

E-string spectrum. Holomorphic anomaly enters via the dependence on E_2 of Z_n and is governed by the relation [12]

$$\frac{\partial Z_n}{\partial E_2} = \frac{1}{24}\sum_{m=1}^{n-1} m(n-m) Z_m Z_{n-m} \ . \tag{3.1}$$

(3.1) corresponds to the reduction of the $\mathrm{U}(n)$ connection down to that of $\mathrm{U}(m)\times\mathrm{U}(n-m)$. Holomorphic anomaly is tied with the fact that $\frac{1}{2}K_3$ does not possess a holomorphic 2-form and there exists no mass perturbation of $\mathcal{N}=4$ theory which splits the location of n M5-branes. Reduction of supersymmetry leads to a binding force among the branes.

Gap condition on the other hand requires that instantons with degree $k<n$ do not exist in $\mathrm{U}(n)$ gauge theory. This comes from the positivity of intersection numbers among holomorphic curves in $\frac{1}{2}K_3$.

These two conditions and consideration of modular invariance uniquely determine the amplitudes. We list the first 3 instanton amplitudes rewritten slightly from [3]

$$Z_1 = \frac{P(m_i;\tau)}{\eta^{12}} \tag{3.2}$$

where

$$P(m_i;\tau) = \frac{1}{2}\left[\sum_{\ell=1}^{4}\prod_{j=1}^{8}\theta_\ell(m_j|\tau)\right], \tag{3.3}$$

$$\begin{aligned} Z_2 &= \frac{1}{\eta^{24}}\Big[f_{20}(\tau)P(2m_i;2\tau)+f_{21}(\tau)P\Big(m_i;\frac{\tau}{2}\Big)+f_{21}(\tau+1)P\Big(m_i;\frac{\tau+1}{2}\Big)\Big]+\\ &\quad+\frac{1}{24}E_2{Z_1}^2 \end{aligned}\tag{3.4}$$

where

$$f_{20}(\tau) = \frac{1}{24}\theta_3(\tau)^4\theta_4(\tau)^4\left(\theta_3(\tau)^4+\theta_4(\tau)^4\right), \tag{3.5}$$

$$f_{21}(\tau) = -\frac{1}{384}\theta_3(\tau)^4\theta_2(\tau)^4\left(\theta_3(\tau)^4+\theta_2(\tau)^4\right), \tag{3.6}$$

$$\begin{aligned} Z_3 &= \frac{1}{\eta^{36}}\left[f_{30}(\tau)P(3m_i;3\tau)+f_{31}(\tau)P\left(m_i;\frac{\tau}{3}\right)+f_{31}(\tau+1)P\left(m_i;\frac{\tau+1}{3}\right)+\right.\\ &\quad\left.+f_{31}(\tau+2)P\left(m_i;\frac{\tau+2}{3}\right)-\frac{1}{288}E_4(\tau)P(m_i;\tau)^3\right]+\frac{1}{6}E_2Z_2Z_1-\frac{1}{288}{E_2}^2{Z_1}^3 \end{aligned}\tag{3.7}$$

where

$$f_{30}(\tau) = \frac{5}{216}\frac{\eta(\tau)^{36}}{\eta(3\tau)^{12}}+\frac{9}{8}\eta(\tau)^{24}, \tag{3.8}$$

$$f_{31}(\tau) = \frac{5}{24}\frac{\eta(\tau)^{36}}{\eta(\tau/3)^{12}}+\frac{1}{72}\eta(\tau)^{24}. \tag{3.9}$$

Note that the function $P(m_i;\tau)$ is (up to η^8) the character of the level-one representation of $\widehat{E}_8$ and $P(2m_i;2\tau), P(m_i;\tau/2), P(m_i;\frac{\tau+1}{2})$ are linear combinations of characters of 3 level-two representations of $\widehat{E}_8$. In general the amplitude Z_n is expressed in terms of level-n representations of $\widehat{E}_8$. There exist 1, 3, 5, 10, 15, 27 distinct representations of $\widehat{E}_8$ at the levels 1, 2, 3, 4, 5, 6, respectively. Coefficient functions f_{n0} are modular forms of $\Gamma_1(n)$ which consists of matrices of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $ad-bc=1$ and $a,d=1$, $c=0 \bmod n$. Functions f_{n1} are the S-transform of f_{n0}.

Using holomorphic anomaly and the gap condition we have obtained the partition functions up to Z_8.

4. Six-dimensional curve

Now by using these data we can determine the coefficient functions of SW curve. We present the first few terms and relegate the rest to appendix A.

$$\begin{aligned} b_1 &= -4\frac{P(m_i;\tau)}{E_4}\,, && (4.1)\\ a_2 &= \frac{1}{E_4\Delta}\left[f_{a2,0}(\tau)P(2m_i;2\tau)+f_{a2,1}(\tau)P\left(m_i;\frac{\tau}{2}\right)+f_{a2,1}(\tau+1)P\left(m_i;\frac{\tau+1}{2}\right)\right] && (4.2)\end{aligned}$$

where

$$\begin{aligned} f_{a2,0}(\tau) &= \frac{2}{3}\left(E_4(\tau)-9\theta_2(\tau)^8\right), && (4.3)\\ f_{a2,1}(\tau) &= \frac{1}{24}\left(E_4(\tau)-9\theta_4(\tau)^8\right), && (4.4)\end{aligned}$$

$$b_2 = \frac{1}{{E_4}^2\Delta}\left[f_{b2,0}(\tau)P(2m_i;2\tau)+f_{b2,1}(\tau)P\left(m_i;\frac{\tau}{2}\right)+f_{b2,1}(\tau+1)P\left(m_i;\frac{\tau+1}{2}\right)\right] \quad (4.5)$$

where

$$\begin{aligned} f_{b2,0}(\tau) &= \frac{1}{36}\left(\theta_3(\tau)^4+\theta_4(\tau)^4\right)\left(E_4(\tau)^2+60E_4(\tau)\theta_2(\tau)^8-45\theta_2(\tau)^{16}\right), && (4.6)\\ f_{b2,1}(\tau) &= -\frac{1}{576}\left(\theta_3(\tau)^4+\theta_2(\tau)^4\right)\left(E_4(\tau)^2+60E_4(\tau)\theta_4(\tau)^8-45\theta_4(\tau)^{16}\right), && (4.7)\end{aligned}$$

$$\begin{aligned} a_3 = \frac{1}{E_4^2\Delta^2}\Bigg[& f_{a3,0}(\tau)P(3m_i;3\tau)+f_{a3,1}(\tau)P\left(m_i;\frac{\tau}{3}\right)+f_{a3,1}(\tau+1)P\left(m_i;\frac{\tau+1}{3}\right)+\\ &+f_{a3,1}(\tau+2)P\left(m_i;\frac{\tau+2}{3}\right)+\frac{2}{3}E_6(\tau)P(m_i;\tau)^3\Bigg], \qquad (4.8)\end{aligned}$$

where

$$\begin{aligned} f_{a3,0}(\tau) &= \frac{1}{3}E_4(\tau)h_2(\tau)^2\left(7E_4(\tau)-9h_0(\tau)^4\right), && (4.9)\\ f_{a3,1}(\tau) &= -\frac{1}{3^8}E_4(\tau)h_3\left(\frac{\tau}{3}\right)^2\left(7E_4(\tau)-h_0\left(\frac{\tau}{3}\right)^4\right), && (4.10)\end{aligned}$$

$$\begin{aligned} b_3 = \frac{1}{{E_4}^3\Delta^2}\Bigg[& f_{b3,0}(\tau)P(3m_i;3\tau)+f_{b3,1}(\tau)P\left(m_i;\frac{\tau}{3}\right)+f_{b3,1}(\tau+1)P\left(m_i;\frac{\tau+1}{3}\right)+\\ &+f_{b3,1}(\tau+2)P\left(m_i;\frac{\tau+2}{3}\right)+\frac{1}{54}\left(8{E_4}^3-5{E_6}^2\right)P(m_i;\tau)^3\Bigg], \qquad (4.11)\end{aligned}$$

where

$$f_{b3,0}(\tau) = \frac{1}{18}E_4(\tau)^2 h_2(\tau)^2 \left(32h_2(\tau)^2 + 48h_2(\tau)h_0(\tau)^3 - 81h_0(\tau)^6\right), \tag{4.12}$$

$$f_{b3,1}(\tau) = \frac{1}{2\cdot 3^{12}}E_4(\tau)^2 h_3\left(\frac{\tau}{3}\right)^2 \left(32h_3\left(\frac{\tau}{3}\right)^2 + 48h_3\left(\frac{\tau}{3}\right)h_0\left(\frac{\tau}{3}\right)^3 - 81h_0\left(\frac{\tau}{3}\right)^6\right). \tag{4.13}$$

Here functions h_i are defined by

$$h_0(\tau) = \sum_{n_1,n_2=-\infty}^{\infty} q^{n_1^2+n_2^2-n_1n_2} = \theta_3(2\tau)\theta_3(6\tau) + \theta_2(2\tau)\theta_2(6\tau), \tag{4.14}$$

$$h_2(\tau) = \frac{\eta(\tau)^9}{\eta(3\tau)^3}, \qquad h_3(\tau) = 27\frac{\eta(3\tau)^9}{\eta(\tau)^3}. \tag{4.15}$$

It is fortunate that the six-dimensional curve can be written in a relatively compact expression. As we have mentioned, coefficient functions a_j, b_j are expressed in terms of affine E_8 characters at level-j. Functions $f_{*n,0}$ are modular forms of $\Gamma_1(n)$ and $f_{*n,1}$ are their S-transform. We have checked that our curve reduces to the results of [6] when six of the mass parameters are set equal to zero.

5. Five-dimensional curve

Let us next see how our result reproduces the five-dimensional curve in the limit $q \to 0$. Affine E_8 characters are reduced to those of the representations of finite-dimensional E_8 algebra in this limit. Let us first introduce a labeling of the fundamental representations of E_8 as in figure 1 where we also show the level of each representation. We then introduce a notation

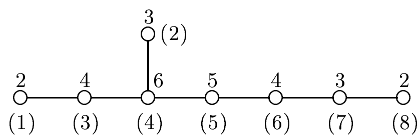

Figure 1: Dynkin diagram for E_8: numbers attached to nodes denote the levels of representations and the numbers in parentheses show their labels.

$$\rho = \begin{bmatrix} & & & n_2 & & & \\ n_1 & n_3 & n_4 & n_5 & n_6 & n_7 & n_8 \end{bmatrix} = \sum_{i=1}^{8} n_i \rho_i, \qquad n_i = 0,1,2,\ldots \tag{5.1}$$

for a representation ρ of E_8 whose highest weight is given by the sum of highest weights of the fundamental representation ρ_i with multiplicity n_i. The level of the representation ρ is given by $\sum_i \ell_i n_i$ where ℓ_i is the level of ρ_i. Weyl-orbit character for the representation ρ is given by

$$w_\rho(m_i) = \sum_{\vec{\alpha}\in\Lambda(\rho)} e^{i\vec{m}\cdot\vec{\alpha}} \tag{5.2}$$

where $\vec{\alpha}$ runs over the weights $\Lambda(\rho)$ of the E_8 Weyl orbit containing the highest-weight state of the representation ρ.

With this preparation we can present the five-dimensional E_8 curve

$$\begin{aligned} y^2 &= 4x^3 - F(u;m_i)x - G(u;m_i), \\ F(u;m_i) &= A_0u^4 + A_2(m_i)u^2 + A_3(m_i)u + A_4(m_i), \\ G(u;m_i) &= B_0u^6 + B_1u^5 + B_2(m_i)u^4 + B_3(m_i)u^3 + B_4(m_i)u^2 + B_5(m_i)u + B_6(m_i) \end{aligned} \tag{5.3}$$

with

$$A_0 = \frac{1}{12}\,, \tag{5.4}$$

$$A_2 = -\frac{2}{3}w[\substack{0\\1000000}] + 12w[\substack{0\\0000001}] - 1440\,, \tag{5.5}$$

$$A_3 = -2w[\substack{1\\0000000}] + 96w[\substack{0\\1000000}] - 1152w[\substack{0\\0000001}] + 103680\,, \tag{5.6}$$

$$\begin{aligned} A_4 = {}& \frac{4}{3}w[\substack{0\\2000000}] - \frac{4}{3}w[\substack{0\\0100000}] - \frac{8}{3}w[\substack{0\\0000100}] - \frac{16}{3}w[\substack{0\\1000001}] + 24w[\substack{0\\0000002}] + \\ & + \frac{328}{3}w[\substack{1\\0000000}] + 48w[\substack{0\\0000010}] - \frac{8960}{3}w[\substack{0\\1000000}] + 28128w[\substack{0\\0000001}] - 2105280\,, \end{aligned} \tag{5.7}$$

$$B_0 = \frac{1}{216}\,, \tag{5.8}$$

$$B_1 = -4\,, \tag{5.9}$$

$$B_2 = -\frac{1}{18}w[\substack{0\\1000000}] - 3w[\substack{0\\0000001}] + 840\,, \tag{5.10}$$

$$B_3 = -\frac{1}{6}w[\substack{1\\0000000}] - 4w[\substack{0\\0000010}] - 8w[\substack{0\\1000000}] + 528w[\substack{0\\0000001}] - 79680\,, \tag{5.11}$$

$$\begin{aligned} B_4 = {}& \frac{2}{9}w[\substack{0\\2000000}] + \frac{1}{9}w[\substack{0\\0100000}] - \frac{28}{9}w[\substack{0\\0000100}] - \frac{152}{9}w[\substack{0\\1000001}] - \\ & - 92w[\substack{0\\0000002}] - \frac{316}{9}w[\substack{1\\0000000}] + 416w[\substack{0\\0000010}] + \frac{13088}{9}w[\substack{0\\1000000}] - \\ & - 35536w[\substack{0\\0000001}] + 3911520\,, \end{aligned} \tag{5.12}$$

$$\begin{aligned} B_5 = {}& \frac{2}{3}w[\substack{1\\1000000}] - \frac{2}{3}w[\substack{0\\0001000}] - \frac{16}{3}w[\substack{0\\1000010}] - \frac{74}{3}w[\substack{1\\0000001}] - 96w[\substack{0\\0000011}] - \\ & - \frac{160}{3}w[\substack{0\\2000000}] - \frac{280}{3}w[\substack{0\\0100000}] + 80w[\substack{0\\0000100}] + 1088w[\substack{0\\1000001}] + \\ & + 7872w[\substack{0\\0000002}] + \frac{9680}{3}w[\substack{1\\0000000}] - 15264w[\substack{0\\0000010}] - \frac{196448}{3}w[\substack{0\\1000000}] + \\ & + 1075776w[\substack{0\\0000001}] - 97251840\,, \end{aligned} \tag{5.13}$$

$$\begin{aligned} B_6 = {}& -\frac{8}{27}w[\substack{0\\3000000}] + w[\substack{2\\0000000}] + \frac{4}{9}w[\substack{0\\1100000}] + \frac{2}{9}w[\substack{0\\0010000}] - \frac{16}{9}w[\substack{0\\2000001}] - \\ & - \frac{8}{3}w[\substack{1\\0000010}] - 8w[\substack{0\\0000020}] - \frac{88}{9}w[\substack{0\\0100001}] - \frac{112}{3}w[\substack{0\\0000101}] - \frac{1000}{9}w[\substack{0\\1000002}] - \\ & - \frac{992}{3}w[\substack{0\\0000003}] - 70w[\substack{1\\1000000}] - 64w[\substack{0\\0001000}] + \frac{112}{9}w[\substack{0\\1000010}] + 636w[\substack{1\\0000001}] + \\ & + 3472w[\substack{0\\0000011}] + 1584w[\substack{0\\2000000}] + 3038w[\substack{0\\0100000}] + \frac{7376}{9}w[\substack{0\\0000100}] - \\ & - 18480w[\substack{0\\1000001}] - 176608w[\substack{0\\0000002}] - \frac{196384}{3}w[\substack{1\\0000000}] + 197968w[\substack{0\\0000010}] + \\ & + 936200w[\substack{0\\1000000}] - 12291232w[\substack{0\\0000001}] + 971250560. \end{aligned} \tag{5.14}$$

(It is also possible to represent this curve using Weyl-orbit characters only for fundamental representations of E_8. See appendix B. In this case products of characters appear in the formula.)

This curve has a manifest E_8 symmetry and can be derived from the expression of [6] with SO(16) symmetry by a suitable shifting of variables and rearrangement of terms. Note that A_j, B_j contain representations with levels $1 \le l \le j$ (constant term is interpreted as level-one representation). This is consistent with the fact that an affine representation of level j is reduced to finite-dimensional representations with levels $l \leq j$ in the limit $q \to 0$.

In fact we can explicitly show that

$$\lim_{q\to 0} a_j(\tau; m_i) = A_j(m_i)\,, \qquad j = 0,1,2,\ldots,4\,, \tag{5.15}$$

$$\lim_{q\to 0} b_j(\tau; m_i) = B_j(m_i)\,, \qquad j = 0,1,2,\ldots,6 \tag{5.16}$$

hold. Since

$$P(m_i;\tau)\,, \qquad E_4(\tau) \to 1 \qquad \text{as} \qquad q \to 0 \tag{5.17}$$

we can easily check the relation (5.16) for b_1. In general it is convenient to use the basis of affine Weyl-orbit characters which are given by, for instance,

$$\widehat{w}^{(2)}[\substack{0\\0000000}](\tau;m_i) = P(2m_i;2\tau)\,, \tag{5.18}$$

$$\widehat{w}^{(2)}[\substack{0\\1000000}](\tau;m_i) = \frac{1}{2}\left(P\left(m_i;\frac{\tau}{2}\right) + P\left(m_i;\frac{\tau}{2}+\frac{1}{2}\right)\right) - P(2m_i;2\tau)\,, \tag{5.19}$$

$$\widehat{w}^{(2)}[\substack{0\\0000001}](\tau;m_i) = \frac{1}{2}\left(P\left(m_i;\frac{\tau}{2}\right) - P\left(m_i;\frac{\tau}{2}+\frac{1}{2}\right)\right) \tag{5.20}$$

at level-two. Then by taking the limit $q \to 0$ we find

$$\begin{aligned} a_2(\tau;m_i) &= \frac{1}{24E_4(\tau)\Delta(\tau)}\Bigg(\left(-16E_4(\tau) + 9\theta_2(\tau)^8\right)\widehat{w}^{(2)}[\substack{0\\1000000}](\tau;m_i) + \\ &\qquad + 9\left(\theta_3(\tau)^8 - \theta_4(\tau)^8\right)\widehat{w}^{(2)}[\substack{0\\0000001}](\tau;m_i) - \\ &\qquad - 135\theta_2(\tau)^8\widehat{w}^{(2)}[\substack{0\\0000000}](\tau;m_i)\Bigg) \end{aligned} \tag{5.21}$$

$$\to -\frac{2}{3}w[\substack{0\\1000000}](m_i) + 12w[\substack{0\\0000001}](m_i) - 1440 = A_2(m_i)\,, \tag{5.22}$$

$$\begin{aligned} b_2(\tau;m_i) &= \frac{1}{576E_4(\tau)^2\Delta(\tau)} \times \\ &\quad \times\Bigg(\left(\theta_3(\tau)^4 + \theta_4(\tau)^4\right) \times \\ &\qquad \times\left(-16E_4(\tau)^2 - 15E_4(\tau)\theta_2(\tau)^8 + 45\theta_2(\tau)^{16}\right)\widehat{w}^{(2)}[\substack{0\\1000000}](\tau;m_i) + \\ &\qquad + 9\theta_2(\tau)^4\left(-12E_4(\tau)^2 + 25E_4(\tau)\theta_2(\tau)^8 - 15\theta_2(\tau)^{16}\right)\widehat{w}^{(2)}[\substack{0\\0000001}](\tau;m_i) + \\ &\qquad + 135\theta_2(\tau)^8\left(\theta_3(\tau)^4 + \theta_4(\tau)^4\right)\left(7E_4(\tau) - 5\theta_2(\tau)^8\right)\widehat{w}^{(2)}[\substack{0\\0000000}](\tau;m_i)\Bigg) \end{aligned} \tag{5.23}$$

$$\to -\frac{1}{18}w[\substack{0\\1000000}](m_i) - 3w[\substack{0\\0000001}](m_i) + 840 = B_2(m_i)\,. \tag{5.24}$$

Flow of E_8 theory to theories with smaller symmetry groups has been extensively discussed in the literature [14, 15, 6, 16]. In appendix C we present five-dimensional curves with manifest E_n symmetry for $n = 6, 7$.

6. Four-dimensional curve

In order to derive four-dimensional curve we first have to reinstate the radius R of the 5th dimension and redefine the mass parameter as

$$m_i \to R m_i\,, \qquad i = 1, 2, \ldots, 8\,. \tag{6.1}$$

Now the new mass parameters carry the dimension of mass. Four-dimensional limit is obtained at $R \to 0$. Thus we expand the characters w_ρ in power series of R. Lower order terms of the series are canceled if we make a suitable shift of the parameter as

$$u \to u + 6c_2 R^2 - \frac{1}{4}{c_2}^2 R^4 + \frac{1}{240}{c_2}^3 R^6 \tag{6.2}$$

where c_2 is the second order Casimir invariant

$$c_2 = \sum_{i=1}^{8} m_i^2\,. \tag{6.3}$$

When we rescale variables as

$$x \to R^{10} x\,, \qquad y \to R^{15} y\,, \qquad u \to R^6 u\,, \tag{6.4}$$

all the terms in the curve cancel up to order R^{29} and the E_8 curve of 4-dimensional theory [10, 17] appears as the coefficient of R^{30}.

Flow of E_8 theory to theories with lower symmetry has been discussed in [9, 10].

7. Discussions

In this paper we have constructed the six-dimensional Seiberg-Witten curve which amounts to an equation describing the complex structure of $\frac{1}{2}K_3$: by applying the mirror symmetry technique one can generate the number of holomorphic curves and gauge theory partition functions on $\frac{1}{2}K_3$ at any higher order. It will be interesting to see if a similar construction of SW curves is possible for manifolds other than the $\frac{1}{2}K_3$ surface.

In [5] it was suggested that $\mathrm{SL}(2,\mathbf{Z})$ symmetry of $N_f = 4$ four-dimensional SW theory should be identified as the $\mathrm{SL}(2,\mathbf{Z})$ symmetry of the torus T^2 of six-dimensional theory. Thus there must be a way to obtain four-dimensional $N_f = 4$ theory directly from the six-dimensional theory without going through the degenerate limit of T^2 as described in the previous sections. It is interesting to see if there in fact exists such a direct link between four- and six-dimensional theories.

Acknowledgments

K.S. would like to thank M. Naka, Y. Ohtake, S. Terashima for valuable discussions. Research of T.E. is supported in part by the fund for the Special Priority Area no.707 "Supersymmetry and Unified Theory of Elementary Particles" by the Japan Ministry of Education.

A. Coefficient functions of six-dimensional SW curve

$$
\begin{aligned}
a_4 \;=\; & \frac{1}{E_4\Delta^2}\Bigg[16\theta_4(2\tau)^8 P(4m_i;4\tau) + \frac{1}{256}\theta_2\left(\frac{\tau}{2}\right)^8 P\left(m_i;\frac{\tau}{4}\right) + \\
& + \frac{1}{256}\theta_2\left(\frac{\tau+1}{2}\right)^8 P\left(m_i;\frac{\tau+1}{4}\right) + \frac{1}{256}\theta_2\left(\frac{\tau+2}{2}\right)^8 P\left(m_i;\frac{\tau+2}{4}\right) + \\
& + \frac{1}{256}\theta_2\left(\frac{\tau+3}{2}\right)^8 P\left(m_i;\frac{\tau+3}{4}\right) + \theta_4(2\tau)^8 P\left(2m_i;\tau+\frac{1}{2}\right) + \\
& + E_4 P(2m_i;\tau) - \frac{3}{2}\Delta^2 {a_2}^2 + \frac{3}{8}E_4\Delta a_2 {b_1}^2 - \frac{9}{128}{E_4}^2{b_1}^4\Bigg], \qquad \text{(A.1)}
\end{aligned}
$$

$$
\begin{aligned}
b_4 \;=\; & \frac{1}{E_4\Delta^2}\Bigg[f_{b4,0}(\tau)P(4m_i;4\tau) + f_{b4,1}(\tau)P\left(m_i;\frac{\tau}{4}\right) + \\
& + f_{b4,1}(\tau+1)P\left(m_i;\frac{\tau+1}{4}\right) + f_{b4,1}(\tau+2)P\left(m_i;\frac{\tau+2}{4}\right) + \\
& + f_{b4,1}(\tau+3)P\left(m_i;\frac{\tau+3}{4}\right) + f_{b4,2}(\tau)P\left(2m_i;\tau+\frac{1}{2}\right) + \\
& + \frac{5}{48}E_6 P(2m_i;\tau) - \frac{1}{96}E_4E_6\Delta b_3 b_1 - 3E_4\Delta b_2 {b_1}^2 - \frac{55}{384}E_4E_6{b_1}^4\Bigg] \qquad \text{(A.2)}
\end{aligned}
$$

where

$$
\begin{aligned}
f_{b4,0}(\tau) &= -\frac{8}{9}\left(32\theta_3(2\tau)^{12} - 75\theta_3(2\tau)^4\theta_4(2\tau)^8 + 2\theta_4(2\tau)^{12}\right), && \text{(A.3)}\\
f_{b4,1}(\tau) &= \frac{1}{2^{11}\cdot 9}\left(32\theta_3\left(\frac{\tau}{2}\right)^{12} - 75\theta_3\left(\frac{\tau}{2}\right)^4\theta_2\left(\frac{\tau}{2}\right)^8 + 2\theta_2\left(\frac{\tau}{2}\right)^{12}\right), && \text{(A.4)}\\
f_{b4,2}(\tau) &= -\frac{1}{18}\left(32\theta_2(2\tau)^{12} - 75\theta_2(2\tau)^4\theta_4(2\tau)^8 - 2\theta_4(2\tau)^{12}\right), && \text{(A.5)}
\end{aligned}
$$

$$
\begin{aligned}
b_5 \;=\; & \frac{1}{{E_4}^2\Delta^3}\Bigg[f_{b5,0}(\tau)P(5m_i;5\tau) + f_{b5,1}(\tau)P\left(m_i;\frac{\tau}{5}\right) + f_{b5,1}(\tau+1)P\left(m_i;\frac{\tau+1}{5}\right) + \\
& + f_{b5,1}(\tau+2)P\left(m_i;\frac{\tau+2}{5}\right) + f_{b5,1}(\tau+3)P\left(m_i;\frac{\tau+3}{5}\right) + \\
& + f_{b5,1}(\tau+4)P\left(m_i;\frac{\tau+4}{5}\right) + \\
& + \Bigg(g_{b5,0}(\tau)P(4m_i;4\tau) + g_{b5,1}(\tau)P\left(m_i;\frac{\tau}{4}\right) + \\
& \quad + g_{b5,1}(\tau+1)P\left(m_i;\frac{\tau+1}{4}\right) + g_{b5,1}(\tau+2)P\left(m_i;\frac{\tau+2}{4}\right) + \\
& \quad + g_{b5,1}(\tau+3)P\left(m_i;\frac{\tau+3}{4}\right) + g_{b5,2}(\tau)P\left(2m_i;\tau+\frac{1}{2}\right)\Bigg)P(m_i;\tau)\Bigg] + \\
& + \frac{1}{\Delta^3}\left[\frac{3}{2}\Delta^2 b_3{b_1}^2 + \frac{3}{16}E_6\Delta b_2{b_1}^3 + \frac{1}{9216}(53{E_4}^3 + 67{E_6}^2){b_1}^5\right] \qquad \text{(A.6)}
\end{aligned}
$$

where

$$\begin{aligned} f_{b5,0}(\tau) &= \frac{2}{3}\eta(\tau)^{16}\eta(5\tau)^{16} \times \\ &\quad\times\left(5^{11} + 4\cdot 5^9\frac{\eta(\tau)^6}{\eta(5\tau)^6} + 22\cdot 5^6\frac{\eta(\tau)^{12}}{\eta(5\tau)^{12}} + 4\cdot 5^4\frac{\eta(\tau)^{18}}{\eta(5\tau)^{18}} - 31\frac{\eta(\tau)^{24}}{\eta(5\tau)^{24}}\right), \qquad \text{(A.7)} \\ f_{b5,1}(\tau) &= \frac{2}{3}\eta(\tau)^{16}\eta(\tau/5)^{16} \times \\ &\quad\times\left(\frac{1}{5} + 4\frac{\eta(\tau)^6}{\eta(\tau/5)^6} + 22\frac{\eta(\tau)^{12}}{\eta(\tau/5)^{12}} + 20\frac{\eta(\tau)^{18}}{\eta(\tau/5)^{18}} - 31\frac{\eta(\tau)^{24}}{\eta(\tau/5)^{24}}\right), \qquad \text{(A.8)} \\ g_{b5,0}(\tau) &= \frac{2}{3}\theta_3(2\tau)^4\theta_4(2\tau)^8 \times \qquad \text{(A.9)} \\ &\quad\times\Big(-1024\theta_3(2\tau)^{12} + 1664\theta_3(2\tau)^8\theta_4(2\tau)^4 - 644\theta_3(2\tau)^4\theta_4(2\tau)^8 + 55\theta_4(2\tau)^{12}\Big), \\ g_{b5,1}(\tau) &= \frac{1}{2^{19}\cdot 3}\theta_3\left(\frac{\tau}{2}\right)^4\theta_2\left(\frac{\tau}{2}\right)^8 \times \qquad \text{(A.10)} \\ &\quad\times\left(-1024\theta_3\left(\frac{\tau}{2}\right)^{12} + 1664\theta_3\left(\frac{\tau}{2}\right)^8\theta_2\left(\frac{\tau}{2}\right)^4 - 644\theta_3\left(\frac{\tau}{2}\right)^4\theta_2\left(\frac{\tau}{2}\right)^8 + 55\theta_2\left(\frac{\tau}{2}\right)^{12}\right), \\ g_{b5,2}(\tau) &= -\frac{1}{24}\theta_2(2\tau)^4\theta_4(2\tau)^8 \times \qquad \text{(A.11)} \\ &\quad\times\Big(1024\theta_2(2\tau)^{12} + 1664\theta_2(2\tau)^8\theta_4(2\tau)^4 + 644\theta_2(2\tau)^4\theta_4(2\tau)^8 + 55\theta_4(2\tau)^{12}\Big), \end{aligned}$$

$$\begin{aligned} b_6 = \frac{1}{E_4\Delta^3}\Bigg[& f_{b6,0}(\tau)P(6m_i;6\tau) + f_{b6,1}(\tau)P\left(m_i;\frac{\tau}{6}\right) + f_{b6,1}(\tau+1)P\left(m_i;\frac{\tau+1}{6}\right) + \\ & + f_{b6,1}(\tau+2)P\left(m_i;\frac{\tau+2}{6}\right) + f_{b6,1}(\tau+3)P\left(m_i;\frac{\tau+3}{6}\right) + \\ & + f_{b6,1}(\tau+4)P\left(m_i;\frac{\tau+4}{6}\right) + f_{b6,1}(\tau+5)P\left(m_i;\frac{\tau+5}{6}\right) + \\ & + f_{b6,2}(\tau)P\left(3m_i;\frac{3\tau}{2}\right) + f_{b6,2}(\tau+1)P\left(3m_i;\frac{3\tau+1}{2}\right) + \\ & + f_{b6,3}(\tau)P\left(2m_i;\frac{2\tau}{3}\right) + f_{b6,3}(\tau+1)P\left(2m_i;\frac{2\tau+2}{3}\right) + \\ & + f_{b6,3}(\tau+2)P\left(2m_i;\frac{2\tau+1}{3}\right)\Bigg] + \\ + \frac{a_2}{E_4\Delta^2}\Bigg[& g_{b6,0}(\tau)P(4m_i;4\tau) + g_{b6,1}(\tau)P\left(m_i;\frac{\tau}{4}\right) + \\ & + g_{b6,1}(\tau+1)P\left(m_i;\frac{\tau+1}{4}\right) + g_{b6,1}(\tau+2)P\left(m_i;\frac{\tau+2}{4}\right) + \\ & + g_{b6,1}(\tau+3)P\left(m_i;\frac{\tau+3}{4}\right) + g_{b6,2}(\tau)P\left(2m_i;\tau+\frac{1}{2}\right)\Bigg] + \\ + \frac{1}{\Delta^3}\Bigg[& -\frac{83}{7344}E_6\Delta^2 b_5 b_1 + \frac{83}{408}\Delta^2 b_4 {b_1}^2 + \frac{1}{34}\Delta^2 b_3 b_2 b_1 + \frac{29}{6528}E_6\Delta b_3 {b_1}^3 + \\ & + \frac{669}{1088}\Delta b_2 {b_1}^4 - \frac{5}{7344}E_4\Delta^2 a_3 a_2 b_1 - \frac{419}{235008}{E_4}^2\Delta a_3 {b_1}^3 - \qquad \text{(A.12)} \\ & - \frac{1}{36864}E_4 E_6\Delta {a_2}^2 {b_1}^2 + \frac{1}{1536}{E_4}^2\Delta a_2 b_2 {b_1}^2 - \frac{1}{256}E_6\Delta {b_2}^2 {b_1}^2 + \frac{1215}{69632}E_6 {b_1}^6\Bigg] \end{aligned}$$

where

$$
\begin{aligned}
f_{b6,0}(\tau) &= -\frac{4}{17}\left(h_0(\tau)+h_0(2\tau)\right)\left(h_0(\tau)-2h_0(2\tau)\right)^2\times \\
&\quad\times\left(27h_0(\tau)^3+84h_0(\tau)^2h_0(2\tau)+72h_0(\tau)h_0(2\tau)^2-32h_0(2\tau)^3\right), \qquad \text{(A.13)}\\
f_{b6,1}(\tau) &= \frac{1}{297432}\left(2h_0\left(\frac{\tau}{3}\right)+h_0\left(\frac{\tau}{6}\right)\right)\left(h_0\left(\frac{\tau}{3}\right)-h_0\left(\frac{\tau}{6}\right)\right)^2\times \qquad \text{(A.14)}\\
&\quad\times\left(27h_0\left(\frac{\tau}{3}\right)^3+42h_0\left(\frac{\tau}{3}\right)^2h_0\left(\frac{\tau}{6}\right)+18h_0\left(\frac{\tau}{3}\right)h_0\left(\frac{\tau}{6}\right)^2-4h_0\left(\frac{\tau}{6}\right)^3\right),\\
f_{b6,2}(\tau) &= \frac{1}{136}\left(-2h_0(\tau)+h_0\left(\frac{\tau}{2}\right)\right)\left(h_0(\tau)+h_0\left(\frac{\tau}{2}\right)\right)^2\times\\
&\quad\times\left(27h_0(\tau)^3-42h_0(\tau)^2h_0\left(\frac{\tau}{2}\right)+18h_0(\tau)h_0\left(\frac{\tau}{2}\right)^2+4h_0\left(\frac{\tau}{2}\right)^3\right), \qquad \text{(A.15)}\\
f_{b6,3}(\tau) &= \frac{4}{37179}\left(h_0\left(\frac{\tau}{3}\right)-h_0\left(\frac{2\tau}{3}\right)\right)\left(h_0\left(\frac{\tau}{3}\right)+2h_0\left(\frac{2\tau}{3}\right)\right)^2\times \qquad \text{(A.16)}\\
&\quad\times\left(27h_0\left(\frac{\tau}{3}\right)^3-84h_0\left(\frac{\tau}{3}\right)^2h_0\left(\frac{2\tau}{3}\right)+72h_0\left(\frac{\tau}{3}\right)h_0\left(\frac{2\tau}{3}\right)^2+32h_0\left(\frac{2\tau}{3}\right)^3\right),\\
g_{b6,0}(\tau) &= -\frac{640}{51}\theta_3(2\tau)^4+\frac{32}{9}\theta_4(2\tau)^4, \qquad \text{(A.17)}\\
g_{b6,1}(\tau) &= \frac{5}{408}\theta_3\left(\frac{\tau}{2}\right)^4-\frac{1}{288}\theta_2\left(\frac{\tau}{2}\right)^4, \qquad \text{(A.18)}\\
g_{b6,2}(\tau) &= -\frac{40}{51}\theta_2(2\tau)^4-\frac{2}{9}\theta_4(2\tau)^4. \qquad \text{(A.19)}
\end{aligned}
$$

B. Coefficient functions of 5d SW curve in terms of characters of fundamental representations

$$
\begin{aligned}
A_0 &= \frac{1}{12}\,, \qquad \text{(B.1)}\\
A_2 &= -\frac{2}{3}w_1+12w_8-1440\,, \qquad \text{(B.2)}\\
A_3 &= -2w_2+96w_1-1152w_8+103680\,, \qquad \text{(B.3)}\\
A_4 &= \frac{4}{3}{w_1}^2-4w_3-16w_6-48w_1w_8-144{w_8}^2+\\
&\quad+400w_2+1440w_7+1728w_1+41472w_8-2073600\,, \qquad \text{(B.4)}\\
B_0 &= \frac{1}{216}\,, \qquad \text{(B.5)}\\
B_1 &= -4\,, \qquad \text{(B.6)}\\
B_2 &= -\frac{1}{18}w_1-3w_8+840\,, \qquad \text{(B.7)}\\
B_3 &= -\frac{1}{6}w_2-4w_7-8w_1+528w_8-79680\,, \qquad \text{(B.8)}\\
B_4 &= \frac{2}{9}{w_1}^2-\frac{1}{3}w_3-\frac{16}{3}w_6-24w_1w_8-120{w_8}^2+\\
&\quad+\frac{424}{3}w_2+1272w_7+4608w_1-25920w_8+3939840\,, \qquad \text{(B.9)}
\end{aligned}
$$

$$
\begin{aligned}
B_5 &= \frac{2}{3}w_1w_2 - 4w_5 - 16w_1w_7 + 64w_2w_8 + 288w_7w_8 - 96{w_1}^2 - 60w_3 - 160w_6 + \\
&\quad + 3456{w_8}^2 + 800w_2 - 24480w_7 - 108480w_1 + 933120w_8 - 97873920\,, \qquad \text{(B.10)} \\
B_6 &= -\frac{8}{27}{w_1}^3 + {w_2}^2 + \frac{4}{3}w_1w_3 - 4w_4 - \frac{32}{3}w_1w_6 - 48{w_1}^2w_8 + 48w_2w_7 + 288{w_7}^2 - \\
&\quad - 40w_3w_8 - 480w_6w_8 - 2592w_1{w_8}^2 - 9792{w_8}^3 + \frac{1124}{3}w_1w_2 + 548w_5 + \\
&\quad + 6688w_1w_7 + 1884w_2w_8 + 25632w_7w_8 + 24576{w_1}^2 + 12920w_3 + 88320w_6 + \\
&\quad + 578688w_1w_8 + 1714176{w_8}^2 - 1694400w_2 - 8460000w_7 - 30102720w_1 - \\
&\quad - 104198400w_8 + 721612800 \qquad \text{(B.11)}
\end{aligned}
$$

where $w_1, \ldots, w_8$ denote E_8 Weyl-orbit characters of fundamental representations and are defined as

$$
\begin{aligned}
w_1 &= w\left[{0 \atop 1000000}\right], & w_2 &= w\left[{1 \atop 0000000}\right], & w_3 &= w\left[{0 \atop 0100000}\right], & w_4 &= w\left[{0 \atop 0010000}\right], \\
w_5 &= w\left[{0 \atop 0001000}\right], & w_6 &= w\left[{0 \atop 0000100}\right], & w_7 &= w\left[{0 \atop 0000010}\right], & w_8 &= w\left[{0 \atop 0000001}\right]. \qquad \text{(B.12)}
\end{aligned}
$$

C. Five-dimensional E_n SW curve

E_7 curve:

$$
\begin{aligned}
y^2 &= 4x^3 - \left(\frac{1}{12}u^4 + (-\frac{2}{3}w_1 + 12)u^2 - 2w_2u + (\frac{4}{3}{w_1}^2 - 4w_3 - 16w_6 - 48w_1 - 144)\right)x - \\
&\quad - \left(\frac{1}{216}u^6 + \left(-\frac{1}{18}w_1 - 3\right)u^4 + \left(-\frac{1}{6}w_2 - 4w_7\right)u^3 + \right. \\
&\qquad + \left(\frac{2}{9}{w_1}^2 - \frac{1}{3}w_3 - \frac{16}{3}w_6 - 24w_1 - 120\right)u^2 + \\
&\qquad + \left(\frac{2}{3}w_1w_2 - 4w_5 - 16w_1w_7 + 64w_2 + 288w_7\right)u + \\
&\qquad + \left(-\frac{8}{27}{w_1}^3 + {w_2}^2 + \frac{4}{3}w_1w_3 - 4w_4 - \frac{32}{3}w_1w_6 - 48{w_1}^2 + \right. \\
&\qquad\quad \left.\left. + 48w_2w_7 + 288{w_7}^2 - 40w_3 - 480w_6 - 2592w_1 - 9792\right)\right) \qquad \text{(C.1)}
\end{aligned}
$$

where $w_1, \ldots, w_7$ denote E_7 Weyl-orbit characters and are defined as

$$
\begin{aligned}
w_1 &= w\left[{0 \atop 100000}\right], & w_2 &= w\left[{1 \atop 000000}\right], & w_3 &= w\left[{0 \atop 010000}\right], & w_4 &= w\left[{0 \atop 001000}\right], \\
w_5 &= w\left[{0 \atop 000100}\right], & w_6 &= w\left[{0 \atop 000010}\right], & w_7 &= w\left[{0 \atop 000001}\right]. \qquad \text{(C.2)}
\end{aligned}
$$

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E_6 curve:

$$
\begin{aligned}
y^2 = 4x^3 &- \left(\frac{1}{12}u^4 - \frac{2}{3}w_1 u^2 - 2w_2 u + (\frac{4}{3}{w_1}^2 - 4w_3 - 16w_6)\right) x - \\
&- \left(\frac{1}{216}u^6 - \frac{1}{18}w_1 u^4 + \left(-\frac{1}{6}w_2 - 4\right) u^3 + \left(\frac{2}{9}{w_1}^2 - \frac{1}{3}w_3 - \frac{16}{3}w_6\right) u^2 + \right.\\
&\quad + \left(\frac{2}{3}w_1 w_2 - 4w_5 - 16w_1\right) u + \\
&\quad \left. + \left(-\frac{8}{27}{w_1}^3 + {w_2}^2 + \frac{4}{3}w_1 w_3 - 4w_4 - \frac{32}{3}w_1 w_6 + 48w_2 + 288\right)\right)
\end{aligned}
\tag{C.3}
$$

where $w_1, \ldots, w_6$ denote E_6 Weyl-orbit characters and are defined as

$$
\begin{aligned}
&w_1 = w[\substack{0\\10000}]\,, \qquad w_2 = w[\substack{1\\00000}]\,, \qquad w_3 = w[\substack{0\\01000}]\,, \qquad w_4 = w[\substack{0\\00100}]\,, \\
&w_5 = w[\substack{0\\00010}]\,, \qquad w_6 = w[\substack{0\\00001}]\,.
\end{aligned}
\tag{C.4}
$$

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